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Bull. Sci. math. 136 (2012) 172–200

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

# $p$-Adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function

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Received 30 May 2011

Available online 5 July 2011

**Abstract**

Let $\mathbb{K}$ be a complete algebraically closed $p$-adic field of characteristic zero. Let $f$, $g$ be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let $P$ be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or in an open disk and let $\alpha$ be a small meromorphic function with regards to $f$ and $g$. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ counting multiplicity, then we show that $f = g$ provided that the multiplicity order of zeroes of $P'$ satisfy certain inequalities. If $\alpha$ is a Moebius function or a non-zero constant, we can obtain more general results on $P$.



*MSC:* 12J25; 30D35; 30G06

*Keywords:* Meromorphic; Nevanlinna; Ultrametric; Sharing value; Unicity; Distribution of values

## 1. Introduction and main results

Let $f$, $g$ be two meromorphic functions in a $p$-adic field. Here we study polynomials $P$ such that, when $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha$, then $f = g$. Problems of uniqueness

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1 Partially funded by the research project CONICYT (“Inserción de nuevos investigadores en la academia”, No. 79090014) from the Chilean Government.

0007-4497/$ – see front matter 
doi:10.1016/j.bulsci.2011.06.006

on meromorphic functions were examined first in $\mathbb{C}$ in [7,8,10,13–17,22,23] and next in a $p$-adic field in [1,3–5,11,12,18,20,21]. After examining problems of the form $P(f) = P(g)$, several studies were made on the equality $f'P'(f) = g'P'(g)$, or value sharing questions: if $f'P'(f)$ and $g'P'(g)$ share a value, or a small function, do we have $f = g$? Here we will try to generalize results previously obtained no matter what the number of zeroes of $P'$. Moreover results also apply to meromorphic functions inside an open disk.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $|\,.\,|$. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

Let $a \in \mathbb{K}$ and $R \in ]0, +\infty[$. We denote by $d(a, R)$ the closed disk $\{x \in \mathbb{K}\colon |x - a| \leqslant R\}$ and by $d(a, R^-)$ the "open" disk $\{x \in \mathbb{K}\colon |x - a| < R\}$. We denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n\in\mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

The problem of value sharing a small function by functions of the form $f'P'(f)$ was examined first when $P$ was just of the form $x^n$ in [7,18,24]. More recently it was examined when $P$ was a polynomial such that $P'$ had exactly two distinct zeroes in [15,17,20], both in complex analysis and in $p$-adic analysis. In [15,17] the functions where meromorphic on $\mathbb{C}$, with a small function that was a constant or the identity. In [20], the problem was considered for analytic functions in the field $\mathbb{K}$: on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Actually solving a value sharing problem involving $f'P'(f)$, $g'P'(g)$ requires to know polynomials of uniqueness $P$ for meromorphic functions.

In [20] the third author studied several problems of uniqueness and particularly the following: *Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. Let $f, g \in \mathcal{A}_u(d(0, R^-))$) and $\alpha \in \mathcal{A}(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_u(d(0, R^-))$) be a small function, such that $f^n(f - a)^k f'$ and $g^n(g - a)^k g'$ share $\alpha$, counting multiplicity, with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$* (see Theorems D and E below).

Here we consider functions $f, g \in \mathcal{M}(\mathbb{K})$ or $f, g \in \mathcal{M}(d(a, R^-))$ and ordinary polynomials $P$: we must only assume certain hypotheses on the multiplicity order of the zeroes of $P'$. The method for the various theorems we will show is the following: assuming that $f'P'(f)$ and $g'P'(g)$ share a small function, we first prove that $f'P'(f) = g'P'(g)$. Next, we derive $P(f) = P(g)$. And then, when $P$ is a polynomial of uniqueness for the functions we consider, we can conclude $f = g$.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field $\mathbb{K}$ and a few specific properties of ultrametric analytic or meromorphic functions.

Let log be a real logarithm function of base $b > 1$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in ]0, +\infty[$ (resp. $r \in ]0, R[$) and let $\gamma \in d(0, r)$. If $f$ has a zero of order $n$ at $\gamma$, we put $\omega_\gamma(f) = n$. If $f$ has a pole of order $n$ at $\gamma$, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r, f)$ the *counting function of zeroes of* $f$ in $d(0, r)$, counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f)>0,\, |\gamma|\leqslant r} \omega_\gamma(f)\bigl(\log r - \log|\gamma|\bigr).$$

Similarly, we denote by $\overline{Z}(r, f)$ the counting function of zeroes of $f$ in $d(0, r)$, ignoring multiplicity, and set

$$\overline{Z}(r, f) = \sum_{\omega_\gamma(f)>0,\, |\gamma|\leqslant r} \bigl(\log r - \log|\gamma|\bigr).$$

In the same way, we set $N(r, f) = Z(r, \frac{1}{f})$ (resp. $\overline{N}(r, f) = \overline{Z}(r, \frac{1}{f})$) to denote the *counting function of poles of* $f$ in $d(0, r)$, counting multiplicity (resp. ignoring multiplicity).

For $f \in \mathcal{M}(d(0, R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r, f) = \max\{Z(r, f) + \log|f(0)|, N(r, f)\}$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

**Definition.** Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0 is called *a small function with respect to* $f$, if it satisfies $\lim_{r\to+\infty} \frac{T(r,\alpha)}{T(r,f)} = 0$ (resp. $\lim_{r\to R^-} \frac{T(r,\alpha)}{T(r,f)} = 0$).

If 0 is a zero or a pole of $f$ or $\alpha$, we can make a change of variable such that the new origin is not a zero or a pole for both $f$ and $\alpha$. Thus it is easily seen that the last relation does not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to $f$ in $\mathbb{K}$ (resp. in $d(0, R^-)$).

**Remark 1.** Thanks to classical properties of the Nevanlinna function $T(r, f)$ with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leqslant T(r, f) + T(r, g)$ and $T(r, fg) \leqslant T(r, f) + T(r, g)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proved that $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$) and that $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ (resp. of $\mathcal{M}_f(d(0, R^-))$) in [6].

Let us remember the following definition.

**Definition.** Let $f, g, \alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))$). We say that $f$ and $g$ *share the function* $\alpha$ *C.M.*, if $f - \alpha$ and $g - \alpha$ have the same zeroes with the same multiplicity in $\mathbb{K}$ (resp. in $d(0, R^-)$).

Recall that a polynomial $P \in \mathbb{K}[x]$ is called *a polynomial of uniqueness* for a class of functions $\mathcal{F}$ if for any two functions $f, g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

The definition of polynomials of uniqueness was introduced in [16] by P. Li and C.C. Yang and was studied in many papers (see [9,10]) for complex functions and in [1,4,5,11,12,14,21], for $p$-adic functions.

Actually, in a $p$-adic field, we can obtain various results, not only for functions defined in the whole field $\mathbb{K}$ but also for functions defined inside an open disk because the $p$-adic Nevanlinna Theory works inside a disk, for functions of $\mathcal{M}_u(d(0, R^-))$.

Let us recall Theorem A in [5,21]:

**Theorem A.** *Let $P \in \mathbb{K}[x]$ be such that $P'$ has exactly two distinct zeroes $\gamma_1$ of order $c_1$ and $\gamma_2$ of order $c_2$. Then $P$ is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{c_1, c_2\} \geqslant 2$, then $P$ is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Theorem A was first proved in [21] with the additional hypothesis $P(c_1) \neq P(c_2)$. Actually this hypothesis is useless because, as showed in Lemma 10 in [5], the equality $P(c_1) = P(c_2)$ is impossible since $P'$ only has two distinct zeroes.

**Notation.** Let $P \in \mathbb{K}[x] \setminus \mathbb{K}$ and let $\Xi(P)$ be the set of zeroes $c$ of $P'$ such that $P(c) \neq P(d)$ for every zero $d$ of $P'$ other than $c$. We denote by $\Phi(P)$ the cardinal of $\Xi(P)$.

**Remark 2.** If $\deg(P) = q$ then $\Phi(P) \leqslant q - 1$.

From [5] we have the following results:

**Theorem B.** *Let $d(a, R^-)$ be an open disk in $\mathbb{K}$ and $P \in \mathbb{K}[x]$. If $\Phi(P) \geqslant 2$ then $P$ is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geqslant 3$ then $P$ is a polynomial of uniqueness for both $\mathcal{A}_u(d(a, R^-))$ and $\mathcal{M}(\mathbb{K})$. If $\Phi(P) \geqslant 4$ then $P$ is a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$.*

And from [20] we have:

**Theorem C.** *Let $P \in \mathbb{K}[x]$ be of degree $n \geqslant 6$ be such that $P'$ only has two distinct zeroes, one of them being of order 2. Then $P$ is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$.*

In [19], the third author proved the following theorems concerning entire functions and analytic functions in a disk:

**Theorem D.** *Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share the function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$. If $n \geqslant \max\{6-k, k+1\}$, then $f = g$. Moreover, if $\alpha \in \mathbb{K} \setminus \{0\}$ and $n \geqslant \max\{5-k, k+1\}$, then $f = g$.*

**Theorem E.** *Let $f, g \in \mathcal{A}_u(d(0, R^-))$, let $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$ and let $a \in \mathbb{K} \setminus \{0\}$. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share the function $\alpha$ C.M. and $n \geqslant 4$, then $f = g$. Moreover, if $f^n(f-a)f'$ and $g^n(g-a)g'$ share the function $\alpha$ C.M. and $n \geqslant 5$, then again $f = g$.*

We can now state our main theorems.

**Theorem 1.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 2$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$, and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,
$n \geqslant k + 2$,
*if* $l = 2$, *then* $n \neq 2k, 2k+1, 3k+1$,
*if* $l = 3$, *then* $n \neq 2k+1, 3k_i - k\ \forall i = 2, 3$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

By Theorem B, we have Corollary 1.1:

**Corollary 1.1.** *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geqslant 3$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 3$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,
$n \geqslant k+2$,
*if* $l = 3$, *then* $n \neq 2k+1, 3k_i - k$ $\forall i = 2, 3$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

And by Theorem A we also have Corollary 1.2.

**Corollary 1.2.** *Let $P \in \mathbb{K}[x]$ be such that $P'$ is of the form $b(x-a_1)^n(x-a_2)^k$ with* $\min(k, n) \geqslant 2$.

*Suppose $P$ satisfies the following conditions*:

$n \geqslant 10 + \max(0, 5-k)$,
$n \geqslant k+2$,
$n \neq 2k, 2k+1, 3k+1$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 2.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 2$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$, and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,
$n \geqslant k+2$,
*if* $l = 2$, *then* $n \neq 2k, 2k+1, 3k+1$,
*if* $l = 3$, *then* $n \neq 2k+1, 3k_i - k$ $\forall i = 2, 3$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

By Theorem B, we have Corollary 2.1.

**Corollary 2.1.** *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geqslant 3$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 3$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$, and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,

$n \geqslant k+2$,
*if* $l=3$, *then* $n \neq 2k+1, 3k_i - k\ \forall i = 2, 3$.

*Let* $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha$ *be a Moebius function. If* $f'P'(f)$ *and* $g'P'(g)$ *share* $\alpha$ *C.M., then* $f = g$.

And by Theorem A, we have Corollary 2.2.

**Corollary 2.2.** *Let* $P \in \mathbb{K}[x]$ *be such that* $P'$ *is of the form* $b(x-a_1)^n(x-a_2)^k$ *with* $\min(k,n) \geqslant 2$ *and with* $b \in \mathbb{K}^*$. *Suppose* $P$ *satisfies the following conditions*:

$n \geqslant 9 + \max(0, 5-k)$,
$n \geqslant k+2$,
$n \neq 2k, 2k+1, 3k+1$.

*Let* $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha$ *be a Moebius function. If* $f'P'(f)$ *and* $g'P'(g)$ *share* $\alpha$ *C.M., then* $f = g$.

**Theorem 3.** *Let* $P$ *be a polynomial of uniqueness for* $\mathcal{M}(\mathbb{K})$, *let* $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $l \geqslant 2$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$, *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose* $P$ *satisfies the following conditions*:

$n \geqslant k+2$,
$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$.

*Let* $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha$ *be a non-zero constant. If* $f'P'(f)$ *and* $g'P'(g)$ *share* $\alpha$ *C.M., then* $f = g$.

By Theorem B, we have Corollary 3.1.

**Corollary 3.1.** *Let* $P \in \mathbb{K}[x]$ *satisfy* $\Phi(P) \geqslant 3$, *let* $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $l \geqslant 3$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l-1$, *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose* $P$ *satisfies the following conditions*:

$n \geqslant k+2$,
$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$.

*Let* $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha$ *be a non-zero constant. If* $f'P'(f)$ *and* $g'P'(g)$ *share* $\alpha$ *C.M., then* $f = g$.

And by Theorem A, we have Corollary 3.2.

**Corollary 3.2.** *Let* $P \in \mathbb{K}[x]$ *be such that* $P'$ *is of the form* $b(x-a_1)^n(x-a_2)^k$ *with* $k \geqslant 2$ *and with* $b \in \mathbb{K}^*$. *Suppose* $P$ *satisfies the following conditions*:

$n \geqslant 9 + \max(0, 5-k)$,
$n \geqslant k+2$,
$n \neq 2k, 2k+1, 3k+1$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 4.** *Let $a \in K$ and $R > 0$. Let $P$ be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ and let $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 2$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l - 1$, and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 10 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$,
$n \geqslant k + 3$,
*if* $l = 2$, *then* $n \neq 2k, 2k + 1, 3k + 1$,
*if* $l = 3$, *then* $n \neq 2k + 1, 3k_i - k\ \forall i = 2, 3$.

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

By Theorem B we can state Corollary 4.1.

**Corollary 4.1.** *Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geqslant 4$, let $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geqslant 4$, $k_i \geqslant k_{i+1}$, $2 \leqslant i \leqslant l - 1$, and let $k = \sum_{i=2}^{l} k_i$. Suppose $P$ satisfies the following conditions*:

$n \geqslant 10 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$,
$n \geqslant k + 3$.

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

And by Theorem C we have Corollary 4.2:

**Corollary 4.2.** *Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ be such that $P'$ is of the form $b(x - a_1)^n (x - a_2)^2$ with $b \in \mathbb{K}^*$. Suppose $P$ satisfies*:

$n \geqslant 13$.

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_f(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 5.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geqslant 3$, $b \in \mathbb{K}^*$, satisfying*:

$n \geqslant l + 10$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

By Theorem B, we have Corollary 5.1:

**Corollary 5.1.** *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geqslant 3$ and be such that $P'$ is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geqslant 3$, $b \in \mathbb{K}^*$ satisfying*:

$$n \geqslant l+10.$$

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 6.** *Let $a \in K$ and $R > 0$. Let $P$ be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ such that $P'$ is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geqslant 3$, $b \in \mathbb{K}^*$ satisfying*:

$$n \geqslant l+10.$$

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

By Theorem B, we have Corollary 6.1:

**Corollary 6.1.** *Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geqslant 4$ and be such that $P'$ is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geqslant 4$, $b \in \mathbb{K}^*$ and $n \geqslant l+10$.*

*Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Example.** Let $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$. Then $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$. We check that:

$$\begin{aligned}
&P(0) = 0,\\
&P(1) = \tfrac{1}{18} - \tfrac{2}{17} - \tfrac{1}{16} + \tfrac{2}{15},\\
&P(-1) = \tfrac{1}{18} + \tfrac{2}{17} - \tfrac{1}{16} - \tfrac{2}{15} \neq 0, P(1), \text{ and}\\
&P(2) = \tfrac{2^{18}}{18} - \tfrac{2^{18}}{17} - \tfrac{2^{16}}{16} + \tfrac{2^{16}}{15} \neq 0, P(1), P(-1).
\end{aligned}$$

Then $\Phi(P) = 4$. So, $P$ is a polynomial of uniqueness for both $\mathcal{M}(\mathbb{K})$ and $\mathcal{M}(d(0, R^-))$. Moreover, we have $n = 14$, $l = 4$, hence we can apply Corollaries 5.1 and 6.1.

Given $f, g \in \mathcal{M}(\mathbb{K})$ transcendental or $f, g \in \mathcal{M}_u(d(0, R^-))$ such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha$ C.M., we have $f = g$.

**Theorem 7.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geqslant 3$, $b \in \mathbb{K}^*$ satisfying*:

$$n \geqslant l+9.$$

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Theorem 8.** *Let $P$ be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that $P'$ is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geqslant 3$, $b \in \mathbb{K}^*$ satisfying*:

$n \geqslant l + 9$.

*Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha$ be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share $\alpha$ C.M., then $f = g$.*

**Example.** Let $P(x) = x^q - ax^{q-2} + b$ with $a \in K^*$, $b \in K$, with $q \geqslant 5$ an odd integer. Then $q$ and $q-2$ are relatively prime and hence by [12, Theorem 3.21] $P$ is a uniqueness polynomial for $\mathcal{M}(\mathbb{K})$ and $P'$ admits 0 as a zero of order $n = q-3$ and two other zeroes of order 1.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}(\mathbb{K})$ be a small function such that $f'P'(f)$, $g'P'(g)$ share $\alpha$ C.M.

Suppose first $q \geqslant 17$. By Theorem 5 we have $f = g$. Now suppose $q \geqslant 15$ and suppose $\alpha$ is a Moebius function or a non-zero constant. Then by Theorems 7 and 8, we have $f = g$.

**Theorem 9.** *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function $\alpha$ C.M. and if $n \geqslant 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{K})$ such that $f = \frac{a(n+2)}{n+1}(\frac{h^{n+1}-1}{h^{n+2}-1})h$ and $g = \frac{a(n+2)}{n+1}(\frac{h^{n+1}-1}{h^{n+2}-1})$. Moreover, if $\alpha$ is a constant or a Moebius function, then the conclusion holds whenever $n \geqslant 11$.*

Inside an open disk, we have a version similar to the general case in the whole field.

**Theorem 10.** *Let $f, g \in \mathcal{M}_u(d(0, R^-))$, and let $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function $\alpha$ C.M. and $n \geqslant 12$, then either $f = g$ or there exists $h \in \mathcal{M}(d(0, R^-))$ such that $f = \frac{a(n+2)}{n+1}(\frac{h^{n+1}-1}{h^{n+2}-1})h$ and $g = \frac{a(n+2)}{n+1}(\frac{h^{n+1}-1}{h^{n+2}-1})$.*

**Remark 3.** In Theorems 9 and 10, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K})$ (resp. let $h \in \mathcal{M}_u(d(0, R^-))$). Now, let us precisely define $f$ and $g$ as: $g = (\frac{n+2}{n+1})(\frac{h^{n+1)}-1}{h^{n+2}-1})$ and $f = hg$. Then we can see that the polynomial $P(y) = \frac{1}{n+2}y^{n+2} - \frac{1}{n+1}y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.

## 2. Basic results

Let us recall a few classical lemmas in [3,4,12]:

**Lemma 1.** *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$), let $a \in \mathbb{K}$, let $Q(x) \in \mathbb{K}[x]$ of degree $s$. Then $T(r, Q(f)) = sT(r, f) + O(1)$ and $T(r, f'Q(f)) \geqslant sT(r, f) + O(1)$.*

**Lemma 2.** *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$). Then $N(r, f') = N(r, f) + \overline{N}(r, f)$, $Z(r, f') \leqslant Z(r, f) + \overline{N}(r, f) - \log r + O(1)$. Moreover, $T(r, f) - Z(r, f) \leqslant T(r, f') - Z(r, f') + O(1)$.*

**Notation.** Given two meromorphic functions $f, g \in \mathcal{M}(K)$ (resp. $f, g \in \mathcal{M}(d(0, R^-))$), we will denote by $\Psi_{f,g}$ the function

$$\frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

**Lemma 3.** *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) and let $\psi = \frac{f'}{f}$. Then $Z(r, \psi) \leqslant N(r, \psi) - \log r + O(1)$.*

The following Lemma 4 is an immediate consequence of Lemma 3:

**Lemma 4.** *The function $\Psi_{f,g}$ satisfies $Z(r, \Psi_{f,g}) \leqslant N(r, \Psi_{f,g}) - \log r$.*

**Lemma 5.** *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f, g \in \mathcal{M}(d(0, R^-))$). If $a$ is a simple zero of $f - 1$ and $g - 1$, it is a zero of $\Psi_{f,g}$.*

In order to state the following lemma, we must recall the definition of quasi-exceptional values in [19].

(i) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$. (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then $b$ will be said to be *a Picard exceptional value* of $f$ (or just *an exceptional value*) if $f(x) \neq b$ $\forall x \in \mathbb{K}$ (resp. $f(x) \neq b$ $\forall x \in d(0, R^-)$).
(ii) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) and let $b \in \mathbb{K}$. Then $b$ will be said to be *a quasi-exceptional value* of $f$ if the function $f - b$ has a finite number of zeroes in $\mathbb{K}$ (resp. in $d(0, R^-)$).

The following results are then immediate in [19]:

**Lemma 6.** *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}$ (resp. let $f \in \mathcal{A}_u(d(0, R^-))$). Then $f$ has no exceptional value. If $f$ is transcendental, it has no quasi-exceptional value. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then $f$ has at most one exceptional value in $\mathbb{K}$. Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then $f$ has at most one quasi-exceptional value in $\mathbb{K}$.*

We now have to recall the *ultrametric Nevanlinna Second Main Theorem* in a basic form which we will frequently use.

Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) satisfy $f'(0) \neq 0, \infty$. Let $S$ be a finite subset of $\mathbb{K}$ and $r \in ]0, +\infty[$ (resp. $r \in ]0, R[$). We denote by $Z_0^S(r, f')$ *the counting function of zeroes of $f'$ in $d(0, r)$ which are not zeroes of any $f - s$ for $s \in S$*. This is, if $(\gamma_n)_{n \in \mathbb{N}}$ is the finite or infinite sequence of zeroes of $f'$ in $d(0, r)$ that are not zeroes of $f - s$ for $s \in S$, with multiplicity order $q_n$ respectively, we set

$$Z_0^S(r, f') = \sum_{|\gamma_n| \leqslant r} q_n \big(\log r - \log |\gamma_n|\big).$$

**Theorem N.** *(See [2,3].) Let $a_1, \dots, a_q \in \mathbb{K}$ with $q \geqslant 2$, $q \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $S = \{a_1, \dots, a_q\}$. Assume that none of $f$, $f'$ and $f - a_j$ with $1 \leqslant j \leqslant q$, equals to $0$ or $\infty$ at the origin. Then, for $r > 0$ (resp. for $r \in ]0, R[$), we have*

$$(q - 1)T(r, f) \leqslant \sum_{j=1}^{q} \overline{Z}(r, f - a_j) + \overline{N}(r, f) - Z_0^S(r, f') - \log r + O(1).$$

## 3. Specific lemmas

We will need the following Lemma 7:

**Lemma 7.** *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geqslant \deg(Q) + 2$ (resp. $n \geqslant \deg(Q) + 3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.*

**Proof.** Put $k = \deg(Q)$. Since $P'(f)f' = P'(g)g'$ there exists $c \in \mathbb{K}$ such that $P(f) = P(g)+c$.

Suppose that $c \neq 0$. Then by Theorem N, we have

$$T\big(r, P(f)\big) \leqslant \overline{Z}\big(r, P(f)\big) + \overline{Z}\big(r, P(f) - c\big) + \overline{N}\big(r, P(f)\big) - \log r + O(1). \tag{1}$$

Obviously we see that $\overline{Z}(r, P(f)) = \overline{Z}(r, f^{n+1}Q(f)) = \overline{Z}(r, fQ(f)) \leqslant T(r, fQ(f))$. By Lemma 1 we have $T(r, fQ(f)) = (k + 1)T(r, f) + O(1)$ and then $\overline{Z}(r, P(f)) \leqslant (k + 1)T(r, f) + O(1)$. We also have $\overline{Z}(r, P(f) - c) = \overline{Z}(r, P(g)) \leqslant \overline{Z}(r, g) + \overline{Z}(r, Q(g)) \leqslant T(r, g) + T(r, Q(g))$. Then by Lemma 1, $\overline{Z}(r, P(f) - c) \leqslant (k + 1)T(r, g) + O(1)$. Notice that $\overline{N}(r, P(f)) = \overline{N}(r, f) \leqslant T(r, f) + O(1)$ then by (1) we obtain

$$T\big(r, P(f)\big) \leqslant (k + 2)T(r, f) + (k + 1)T(r, g) - \log r + O(1). \tag{2}$$

According to Lemma 1 we have $T(r, P(f)) = (n + k + 1)T(r, f) + O(1)$. Then by (2) we have

$$nT(r, f) \leqslant T(r, f) + (k + 1)T(r, g) - \log r + O(1). \tag{3}$$

We similarly we obtain

$$nT(r, g) \leqslant T(r, g) + (k + 1)T(r, f) - \log r + O(1). \tag{4}$$

Hence adding (3) and (4) we have

$$n\big(T(r, f) + T(r, g)\big) \leqslant (k + 2)\big(T(r, f) + T(r, g)\big) - 2\log r + O(1)$$

and then

$$0 \leqslant (k + 2 - n)\big(T(r, f) + T(r, g)\big) - 2\log r + O(1).$$

That leads to a contradiction because $n \geqslant k + 2$ (resp. $n \geqslant k + 3$) and $\lim_{r\to+\infty}(T(r, f) + T(r, g)) = +\infty$ (resp. $\lim_{r\to R}(T(r, f) + T(r, g)) = +\infty$). Thus $c = 0$ and consequently $P(f) = P(g)$. $\square$

**Lemma 8.** *Let $F, G \in \mathcal{M}(\mathbb{K})$ (resp. Let $F, G \in \mathcal{M}(d(0, R^-))$) be non-constant, having no zero and no pole at $0$ and sharing the value $1$ C.M.*

*If $\Psi_{F,G} = 0$ and if*

$$\lim_{r\to+\infty}\big(T(r, F) - \big[\overline{Z}(r, F) + \overline{N}(r, F) + \overline{Z}(r, G) + \overline{N}(r, G)\big]\big) = +\infty$$

(*resp.*

$$\lim_{r\to R^-}\big(T(r, F) - \big[\overline{Z}(r, F) + \overline{N}(r, F) + \overline{Z}(r, G) + \overline{N}(r, G)\big]\big) = +\infty)$$

*then either $F = G$ or $FG = 1$.*

**Proof.** Suppose $\Psi_{F,G} = 0$ and that the above limit is $+\infty$ in the situation we consider: $F, G \in \mathcal{M}(\mathbb{K})$ or $F, G \in \mathcal{M}(d(0, R^-))$.

Since $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = (\frac{F'}{(F-1)^2})(\frac{(G-1)^2}{G'})$, there exist $a, b \in \mathbb{K}$ with $a \neq 0$, such that $\frac{1}{F-1} = \frac{a}{G-1} + b$, this is, $F = \frac{(1+b)G+a-(1+b)}{bG+(a-b)}$. Hence

$$F = \frac{AG+B}{CG+D} \tag{1}$$

with $A, B, C, D \in \mathbb{K}$.

Let $r > 0$ (resp. Let $r \in ]0, R[$). Consider the following three cases:

- **Case 1**: $A \neq 0$ and $C = 0$.
  By (1), we have $F - \frac{B}{D} = \frac{A}{D}G$. Suppose $B \neq 0$. Then $\overline{Z}(r, F - \frac{B}{D}) = \overline{Z}(r, G)$. So, applying Theorem N to $F$, we obtain
  $$\begin{aligned} T(r,F) &\leqslant \overline{Z}(r,F) + \overline{Z}\left(r, F - \frac{B}{D}\right) + \overline{N}(r,F) - \log r + O(1) \\ &= \overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) - \log r + O(1) \\ &< \overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G) + O(1), \end{aligned}$$
  a contradiction to our hypothesis. Thus $B = 0$ and, so $F = \frac{A}{D}G$.
  Suppose $\frac{A}{D} \neq 1$. Since $F$ and $G$ share 1 C.M. and $F = \frac{A}{D}G$, we have $(F(x), G(x)) \neq (1, 1)$ $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(0, R^-)$), because if $F(x) = 1$, then $G(x) = 1$ and hence $\frac{A}{D} = 1$, a contradiction. But $G(x) = 1$ if and only if $F(x) = \frac{A}{D}$. Thus $F$ cannot take values 1 and $\frac{A}{D}$ and hence $F$ has two exceptional values. Consequently, by Lemma 6, $F$ is a constant, a contradiction. Thereby $\frac{A}{D} = 1$, and hence $F = G$.
- **Case 2**: $A = 0$ and $C \neq 0$.
  By (2), we have $G = \frac{B}{CF} - \frac{D}{C}$. Suppose $D \neq 0$. Since $T(r, F) = T(r, \frac{1}{F}) + O(1)$ and $\overline{Z}(r, \frac{1}{F} - \frac{D}{B}) = \overline{Z}(r, G)$, applying Theorem N to $F$, we have
  $$\begin{aligned} T(r,F) &\leqslant \overline{Z}\left(r, \frac{1}{F}\right) + \overline{Z}\left(r, \frac{1}{F} - \frac{D}{B}\right) + \overline{N}\left(r, \frac{1}{F}\right) - \log r + O(1) \\ &= \overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) - \log r + O(1) \\ &< \overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G) + O(1), \end{aligned}$$
  a contradiction to our hypothesis, again. Thus $D = 0$ and, so $F = \frac{B}{CG}$.
  Now, suppose $\frac{B}{C} \neq 1$. Using the same argument as in Case 1, we conclude that $F(x) - 1 \neq 0$ and $G(x) - 1 \neq 0$ $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(0, R^-)$). Moreover, $G(x) = 1$ if and only if $F(x) = \frac{B}{C}$. Then, it is necessary that $F(x) \neq \frac{B}{C}$. Hence, as in Case 1, $F$ omits two values in $\mathbb{K}$ which is impossible (Lemma 6), $F$ is a constant, a contradiction again. Consequently $\frac{B}{C} = 1$ and, hence $FG = 1$.
- **Case 3**: $AC \neq 0$.
  By (1), we have $F - \frac{A}{C} = \frac{B - \frac{AD}{C}}{CG+D}$ and hence $\overline{Z}(r, F - \frac{A}{C}) = \overline{N}(r, G)$. Applying Theorem N to $F$, we have
  $$\begin{aligned} T(r,F) &\leqslant \overline{Z}(r,F) + \overline{Z}\left(r, F - \frac{A}{C}\right) + \overline{N}(r,F) - \log r + O(1) \\ &= \overline{Z}(r,F) + \overline{N}(r,G) + \overline{N}(r,F) - \log r + O(1) \\ &< \overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G) + O(1), \end{aligned}$$
  a contradiction to our hypothesis, again. $\square$

**Lemma 9.** *Let $Q(x) = (x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i} \in \mathbb{K}[x]$ $(a_i \neq a_j,\ \forall i \neq j)$ with $l \geqslant 2$ and $n \geqslant \max\{k_2, \dots, k_l\}$ and let $k = \sum_{i=2}^{l} k_i$. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$) such that $\theta = Q(f) f' Q(g) g'$ is a small function with respect to $f$ and $g$. We have the following:*

*If $l = 2$ then $n$ belongs to $\{k, k+1, 2k, 2k+1, 3k+1\}$.*
*If $l = 3$ then $n$ belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, \dots, 3k_l - k\}$.*
*If $l \geqslant 4$ then $n = k+1$.*
*If $\theta$ is a constant and $f, g \in \mathcal{M}(\mathbb{K})$ then $n = k+1$.*

**Proof.** Without loss of generality, we can assume $a_1 = 0$. Suppose $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$) satisfy

$$f^n\left(\prod_{i=2}^{l}(f - a_i)^{k_i}\right) f' g^n \left(\prod_{i=2}^{l}(g - a_i)^{k_i}\right) g' = \theta. \tag{1}$$

Let $\Sigma$ be the set of zeroes and poles of $\theta$. We will denote by $S(r)$ any function $\varphi$ defined in $]0, +\infty[$ (resp. in $]0, R[$) such that $\lim_{r\to+\infty} \frac{\varphi(r)}{T(r,f)} = 0$ and $\lim_{r\to+\infty} \frac{\varphi(r)}{T(r,g)} = 0$.

Let $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) be a zero of $g$ of order $s$. Clearly, by (1), $\gamma$ is a pole of $f$ of order, for example, $t$. And since $\gamma$ is neither a zero nor a pole of $\theta$ we can derive the following relation

$$s(n+1) = t(n+k+1) + 2. \tag{2}$$

Now, suppose that for $i \in \{2, \dots, l\}$, $g - a_i$ has a zero $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) of order $s_i$. It is a pole of $f$ of order $t_i$. So, by (2), we obtain

$$s_i(k_i + 1) = t_i(n+k+1) + 2. \tag{3}$$

By (2) and (3) it is obvious that $s > t$ and $s_i > t_i$ and hence, $s \geqslant 2$, $s_i \geqslant 2$.

Consider now a pole $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) of $f$. Either it is a zero of $g$, or it is a zero of $g - a_i$ for some $i \in \{2, \dots, l\}$, or it is a zero of $g'$ that is neither a zero of $g$ nor a zero of $g - a_i$ ($\forall i \in \{2, \dots, l\}$). Let $Z_0(r, g')$ be the counting function of zeroes of $g'$ that are neither a zero of $g$ nor a zero of $g - a_i$ for all $i \in \{2, \dots, l\}$ (counting multiplicity) and let $\overline{Z}_0(r, g')$ be the counting function of zeroes of $g'$ that are neither a zero of $g$ nor a zero of $g - a_i$ for all $i \in \{2, \dots, l\}$, ignoring multiplicity. Since $T(r, \theta) = S(r)$, we have

$$\overline{N}(r, f) \leqslant \overline{Z}(r, g) + \sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}_0\big(r, g'\big) + S(r). \tag{4}$$

And if $\theta$ is a constant, we have

$$\overline{N}(r, f) \leqslant \overline{Z}(r, g) + \sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}_0\big(r, g'\big). \tag{5}$$

Now, by Theorem N, we have

$$(l-1)T(r, f) \leqslant \overline{Z}(r, f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{N}(r, f) - Z_0\big(r, f'\big) - \log r + O(1),$$

hence by (4), we obtain

$$(l-1)T(r,f) \leqslant \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}_0(r,g') - Z_0(r,f') + S(r) \tag{6}$$

and if $\theta$ is a constant, by (5) we have

$$(l-1)T(r,f) \leqslant \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}_0(r,g') - Z_0(r,f') - \log r + O(1). \tag{7}$$

And similarly, with $f$, in the general case we have

$$(l-1)T(r,g) \leqslant \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}_0(r,f') - Z_0(r,g') + S(r) \tag{8}$$

and if $\theta$ is a constant, we have

$$(l-1)T(r,g) \leqslant \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}_0(r,f') - Z_0(r,g') - \log r + O(1). \tag{9}$$

Hence, adding (6) and (8), in the general case we obtain

$$(l-1)\big(T(r,f)+T(r,g)\big) \leqslant 2\left(\overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i)\right) + S(r) \tag{10}$$

and if $\theta$ is a constant, by (7) and (9) we have

$$(l-1)\big(T(r,f)+T(r,g)\big) \leqslant 2\left(\overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i)\right) - 2\log r + O(1). \tag{11}$$

**Case $l = 2$.** Without loss of generality, we can assume $a_2 = 1$. Relation (10) now becomes

$$T(r,f) + T(r,g) \leqslant 2\big(\overline{Z}(r,g) + \overline{Z}(r,g-1) + \overline{Z}(r,f) + \overline{Z}(r,f-1)\big) + S(r). \tag{10a}$$

Suppose now that all zeroes of $f$, $f-1$, $g$, $g-1$ are at least of order 5, except maybe those lying in $\Sigma$: then

$$\overline{Z}(r,f)\leqslant\frac{1}{5}T(r,f)+S(r),\qquad \overline{Z}(r,f-1)\leqslant\frac{1}{5}T(r,f)+S(r),$$
$$\overline{Z}(r,g)\leqslant\frac{1}{5}T(r,g)+S(r),\qquad \overline{Z}(r,g-1)\leqslant\frac{1}{5}T(r,g-1)+S(r)$$

a contradiction to (10a), proving the statement of the lemma.

Consequently, we will examine all situations leading to zeroes of order $\leqslant 4$ for $f$, $f-1$, $g$, $g-1$ out of $\Sigma$. Actually, since $f$ and $g$ play the same role with respect to $n$ and $k$, it is sufficient to examine the situation, for instance, when $g$ or $g-1$ has a zero out of $\Sigma$ of order $s\leqslant 4$. In each case we denote by $t$ the order of the pole of $f$ which is a zero of $g$ or $g-1$. Since $s>t$, we only have to examine zeroes of $g$ or $g-1$ that are poles of $f$ of order 1, 2, 3.

Suppose first $g$ has a zero $\gamma\notin\Sigma$ of order $s=2$. Then

$$2(n+1)=t(k+n+1)+2. \tag{12}$$

By (12) if $t=1$ we find a solution:

$$n=k+1. \tag{13}$$

Next, if $t\geqslant 2$, we check that $2n+2<t(k+n+1)+2$, hence (13) is the only solution.

Suppose now $g$ has a zero $\gamma\notin\Sigma$ of order $s=3$. Then

$$3(n+1)=t(k+n+1)+2. \tag{14}$$

By (14) if $t=1$ we find no solution because $k\leqslant n$.

If $t=2$ we find

$$n=2k+1. \tag{15}$$

If $t\geqslant 3$ we have $3(n+3)<3(k+n+1)+2$ hence (15) is the only solution.

Suppose now $g$ has a zero of order $s=4$. Then

$$4(n+1)=t(k+n+1)+2. \tag{16}$$

If $t=1$, since $k\leqslant n$, we have $4(n+1)>t(k+n+1)+2$.

If $t=2$, by (16) we have a solution

$$n=k. \tag{17}$$

If $t=3$, we have another solution

$$n=3k+1. \tag{18}$$

Consequently, by (13), (15), (16), (18), all possibilities for $g$ to have a zero of order $s\leqslant 4$ are as follows:

$$n=k+1,\qquad s=2,\qquad n=2k+1,\qquad s=3,\qquad n=k,\qquad s=4,$$
$$n=3k+1,\qquad s=4. \tag{19}$$

Now, we will examine zeroes of $g-1$ $\gamma$ out of $\Sigma$ of order $\leqslant 4$. So, the order $s'$ of $g-1$ satisfies

$$s'(k+1)=t(k+n+1)+2. \tag{20}$$

Suppose first $g-1$ has a zero $\gamma\notin\Sigma$ of order $s'=2$. Then by (20), we have

$$2(k+1)=t(k+n+1)+2. \tag{21}$$

Since $k \leqslant n$, we find no solution neither when $t = 1$ that would lead to $k = n + 1$, nor when $t \geqslant 2$ because $2(k+1) < t(k+n+1) + 2$.

Suppose now that $s' = 3$.

If $t = 1$ we find a solution:

$$n = 2k. \tag{22}$$

If $t \geqslant 2$, we have no solution with $k \leqslant n$ because $3(k+1) < t(k+n+1) + 2$.

Suppose now that $s' = 4$.

If $t = 1$ we find a solution:

$$n = 3k + 1. \tag{23}$$

If $t = 2$ we find another solution:

$$n = k. \tag{24}$$

If $t \geqslant 3$, we find no solution with $k \leqslant n$ because $4(k+1) < t(k+n+1) + 2$.

Consequently, by (22), (23), (24), all possibilities for $g - 1$ to have a zero $\gamma \notin \Sigma$ of order $s \leqslant 4$ are as follows:

$$n = 2k, \qquad s' = 3, \qquad n = 3k+1, \qquad s' = 4, \qquad n = k, \qquad s' = 4. \tag{25}$$

Thus, we have proved that when $n \neq k, k+1, 2k, 2k+1, 3k+1$, none of the zeroes of $f$, $f - 1$, $g$, $g - 1$ out of $\Sigma$ is of order $\leqslant 4$ and therefore the general statement of the lemma is proved in the case $l = 2$.

Now, suppose that $\theta$ is a constant and $f$, $g$ belong to $\mathcal{M}(K)$ and suppose that $n \neq k + 1$. We notice that $\Sigma$ is now empty. Now (11) gets

$$T(r, f) + T(r, g) \leqslant 2\bigl(\overline{Z}(r, g) + \overline{Z}(r, g-1) + \overline{Z}(r, f) + \overline{Z}(r, f-1)\bigr) - 2\log r + O(1). \tag{11a}$$

First, we have seen that zeroes of order 2 for $g$ or $g - 1$ (hence also for $f$ and $f - 1$) only occur when $n = k + 1$. Consequently, excluding this case, all zeroes of $f$, $f - 1$, $g$, $g - 1$ are of order $\geqslant 3$. We will examine each case.

Suppose first that all zeroes of $f$, $f - 1$, $g$, $g - 1$ are at least of order 4. Then $\overline{Z}(r, f) \leqslant \frac{1}{4}T(r, f)$, $\overline{Z}(r, f-1) \leqslant \frac{1}{4}T(r, f)$, $\overline{Z}(r, g) \leqslant \frac{1}{4}T(r, g)$, $\overline{Z}(r, g-1) \leqslant \frac{1}{4}T(r, g-1)$ a contradiction to (11a).

And finally, suppose that all zeroes of $f$ and $g$ are at least of order 3 and all zeroes of $f - 1$ and $g - 1$ are at least of order 6. Then we have $\overline{Z}(r, f) \leqslant \frac{1}{3}T(r, f)$, $\overline{Z}(r, f-1) \leqslant \frac{1}{6}T(r, f)$, $\overline{Z}(r, g) \leqslant \frac{1}{3}T(r, g)$, $\overline{Z}(r, g-1) \leqslant \frac{1}{6}T(r, g-1)$, a contradiction to (11a) again.

Recall that when $f$ has a pole of order 4, $g$ or $g - 1$, if it has a zero, must have a zero of order $\geqslant 5$. Consequently, we only have to examine zeroes of $g$ or $g - 1$ that are poles of $f$ of order 1, 2, 3.

For each pair $(n, k)$ leading to an order $s > 2$ of zero of $g$, we will precisely examine the possible order of zeroes of $g - 1$ and vice versa.

First we have to consider the case $n = 2k + 1$. We know that if $g$ has a zero, it is at least of order 3. If $g - 1$ has a zero, by (3) its order $s$ satisfies

$$s(k+1) = t(3k+2) + 2. \tag{26}$$

We can check that no solution $(s, t)$ exists with $s \leqslant 4$.

Suppose now $s = 5$.

If $t = 1$, we check that $5(k + 1) > 3k + 4$.

If $t \geqslant 2$, we have $5k + 5 < t(3k + 2) + 2$.

Hence, if $n = 2k + 1$, a zero of $g - 1$ has order $\geqslant 6$. Indeed, such a possibility exists with $s = 6$ and $t = 2$. Consequently, we have

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leqslant \frac{1}{2}T(r, g). \tag{27}$$

Suppose now $n = 3k + 1$. We have seen that all zeroes of $g$ and $g - 1$ have order at least 4. Consequently, we have

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leqslant \frac{1}{2}T(r, g) + O(1). \tag{28}$$

Suppose now $n = k$, then $g$ and $g - 1$ play the same role. All zeroes of $g$ and $g - 1$ are at least of order $\geqslant 4$ hence we have again:

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leqslant \frac{1}{2}T(r, g) + O(1). \tag{29}$$

Finally, suppose $n = 2k$. We have seen that $g$ admits no zero of order $s < 5$. So we must examine the case $s = 5$. By (2) we have $5(2k + 1) = t(2k + k + 1) + 2$. Then $t = 1$ is impossible.

If $t = 2$, we have $10k + 5 = 6k + 4$, impossible.

If $t = 3$, we have $10k + 5 = 9k + 5$.

And if $t > 3$, then $5(2k + 1) < t(3k + 1) + 2$ for all $k \geqslant 2$.

Consequently, all zeroes of $g$ have order at least 6 and hence we have again

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leqslant \frac{1}{2}T(r, g). \tag{30}$$

Thus, since $n \neq k + 1$, by (27), (28), (29), (30) and the symmetric inequalities for $f$ instead of $g$, we have proved a contradiction to (11a).

**Case $l = 3$.** Suppose that all zeroes of $f, g, f - a_i, g - a_i$ $\forall i \in \{2, 3\}$, are at least of order 4, except maybe those lying in $\Sigma$: then

$$\overline{Z}(r, f) \leqslant \frac{1}{4}T(r, f) + S(r) \quad \text{and} \quad \forall i \in \{2, 3\}, \quad \overline{Z}(r, f - a_i) \leqslant \frac{1}{4}T(r, f) + S(r),$$
$$\overline{Z}(r, g) \leqslant \frac{1}{4}T(r, g) + S(r) \quad \text{and} \quad \forall i \in \{2, 3\}, \quad \overline{Z}(r, g - a_i) \leqslant \frac{1}{4}T(r, g) + S(r).$$

Then using (10) we obtain $l \leqslant 2$, a contradiction.

Consequently, we will examine all $n$ and $k_i$ ($i \in \{2, 3\}$) leading to zeroes out of $\Sigma$ of order $\leqslant 3$ for $f, g, f - a_i, g - a_i$ for all $i \in \{2, 3\}$. And since $f$ and $g$ play the same role, it is sufficient to examine the situation, for instance, when $g$ or some $g - a_i$ has a zero of order less than 3. In each case we denote by $t$ the order of the pole of $f$ which is a zero of $g$ or $g - a_i$ for some $i$. Recall that when $f$ has a pole of order 3, $g$ or $g - a_i$, if it has a zero, must have a zero of order $\geqslant 4$. Consequently, we only have to examine zeroes of $g$ or $g - a_i$ ($\forall i \in \{2, 3\}$) that are poles of $f$ of order 1, 2.

Suppose first $g$ has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then by (2) we have

$$2(n + 1) = t(k + n + 1) + 2. \tag{31}$$

By (31) if $t = 1$ we find a solution:

$$n = k + 1. \tag{32}$$

Next, if $t = 2$, we check that $2n + 2 < 2(k + n + 1) + 2$, hence (32) is the only solution.
Suppose now $g$ has a zero $\gamma \notin \Sigma$ of order $s = 3$. Then

$$3(n + 1) = t(k + n + 1) + 2. \tag{33}$$

By (33) if $t = 1$ we find a solution:

$$n = \frac{k}{2}. \tag{34}$$

If $t = 2$ we find

$$n = 2k + 1. \tag{35}$$

Consequently, by (32), (34), (35) all possibilities for $g$ to have a zero of order $s \leqslant 3$ are as follows:

$$n = k + 1, \qquad s = 2, \qquad n = \frac{k}{2}, \qquad s = 3, \qquad n = 2k + 1, \qquad s = 3.$$

Now, let $i \in \{2, 3\}$ and examine zeroes of $g - a_i$, $\gamma \notin \Sigma$ of order $s_i \leqslant 3$. So, by (3), the order $s_i$ of $g - a_i$ satisfies

$$s_i(k_i + 1) = t(k + n + 1) + 2. \tag{36}$$

Suppose first $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then by (36), we have

$$2(k_i + 1) = t(k + n + 1) + 2. \tag{37}$$

Since $k_i \leqslant n$ and $k_i \leqslant k$ we have $2(k_i + 1) < t(k + n + 1) + 2$. Hence we find no solution for (37).

Suppose now $s_i = 3$.
If $t = 1$ we find a solution:

$$3k_i = n + k. \tag{38}$$

If $t = 2$, we have no solution because $3k_i < 2(n + k)$.
Consequently, the unique possibility for $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order $s_i \leqslant 3$ is:

$$n + k = 3k_i, \qquad s_i = 3.$$

Thus, we have proved that when $n \neq k + 1, \frac{k}{2}, 2k + 1, 3k_i - k$ none of the zeroes of $f$, $g$, $f - a_i$, $g - a_i$ ($\forall i \in \{2, 3\}$) out of $\Sigma$ is of order $\leqslant 3$ and therefore the statement of the lemma is proved in the case $l = 3$.

**Case $l \geqslant 4$.** Suppose now that all zeroes of $f$, $g$, $f - a_i$, $g - a_i$ $\forall i \in \{2, \dots, l\}$ are at least of order 3, except maybe those lying in $\Sigma$: then

$$\overline{Z}(r, f) \leqslant \frac{1}{3}T(r, f) + S(r) \quad \text{and} \quad \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, f - a_i) \leqslant \frac{1}{3}T(r, f) + S(r),$$
$$\overline{Z}(r, g) \leqslant \frac{1}{3}T(r, g) + S(r) \quad \text{and} \quad \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, g - a_i) \leqslant \frac{1}{3}T(r, g) + S(r).$$

Then using (10) we obtain $l \leqslant 3$, a contradiction.

Consequently, we will examine all $n$ and $k_i$ ($i \in \{2, \dots, l\}$) leading to zeroes out of $\Sigma$ of order $\leqslant 2$ for $f$, $g$, $f - a_i$, $g - a_i$ for all $i \in \{2, \dots, l\}$. Actually, since $f$ and $g$ play the same role, it is sufficient to examine the situation, for instance, when $g$ or some $g - a_i$ has a zero of order less than 2. In each case we denote by $t$ the order of the pole of $f$ which is a zero of $g$ or $g - a_i$ for some $i$. Recall that when $f$ has a pole of order 2, $g$ or $g - a_i$, if it has a zero, must have a zero of order $\geqslant 3$. And then, we only have to examine zeroes of $g$ or $g - a_i$ ($\forall i \in \{2, \dots, l\}$) that are poles of $f$ of order 1.

Suppose first $g$ has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then $\gamma$ is a pole of $f$ of order $t = 1$. Then by (2) we have

$$2(n+1) = (k+n+1) + 2. \tag{39}$$

We find a solution:

$$n = k+1. \tag{40}$$

Now, let $i \in \{2, \dots, l\}$ and suppose $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then $\gamma$ is a pole of $f$ of order $t = 1$. So by (3) we have:

$$2(k_i + 1) = (n+k+1) + 2.$$

That means $2k_i = n + k + 1$. Since $k_i \leqslant n$ and $k_i \leqslant k$, we find no solution when $s_i = 2$ and $t = 1$.

Consequently, by (40), the only possibility for $g$ or some $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order $\leqslant 2$ is:

$$n = k+1.$$

As in the case $l = 2$, we now have to consider the situation $l \geqslant 3$ when $\theta \in K$ and $f, g$ belong to $\mathcal{M}(K)$.

Obviously $\Sigma = \emptyset$ and we have seen that zeroes of order 2 for $f$, $g$, $f - a_i$, $g - a_i$ ($\forall i \in \{2, \dots, l\}$) only occur when $n = k + 1$. Consequently, excluding this case, all zeroes of $f$, $g$, $f - a_i$, $g - a_i$ ($\forall i \in \{2, \dots, l\}$) are of order $\geqslant 3$.

Thus suppose $n \neq k + 1$. Then,

$$\begin{aligned}
&\overline{Z}(r, f) \leqslant \frac{1}{3}T(r, f) \quad \text{and} \quad \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, f - a_i) \leqslant \frac{1}{3}T(r, f) + O(1), \\
&\overline{Z}(r, g) \leqslant \frac{1}{3}T(r, g) \quad \text{and} \quad \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, g - a_i) \leqslant \frac{1}{3}T(r, g) + O(1).
\end{aligned}$$

By (11), we obtain $(l-1)(T(r, f) + T(r, g)) \leqslant \frac{2l}{3}(T(r, f) + T(r, g)) - 2\log(r) + O(1)$. Hence $l < 3$, a contradiction. This finishes the proof of the lemma. $\square$

Lemma 10 is known and easily checked in [4,23]:

**Lemma 10.** *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f, g \in \mathcal{M}_u(d(0, R^-))$) satisfy $(f - a)f^n = (g - a)g^n$ with $a \in \mathbb{K}$ and let $h = \frac{f}{g}$. If $h$ is not identically 1, then*

$$g = \frac{h^n - 1}{h^{n+1} - 1}, \qquad f = \frac{h^{n+1} - h}{h^{n+1} - 1}.$$

**Notation.** Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) be such that $f(0) \neq 0, \infty$.

We denote by $Z_{1)}(r, f)$ the counting function of simple zeroes of $f$ and by $N_{1)}(r, f)$ the counting function of simple poles of $f$.

We denote by $\overline{Z}_{(2}(r, f)$ the counting function of multiple zeroes of $f$, each counted without multiplicity, and we denote by $N_{(2}(r, f)$ the counting function of multiple poles of $f$, each counted without multiplicity. Consequently, by definition, one has $\overline{Z}(r, f) = Z_{1)}(r, f) + \overline{Z}_{(2}(r, f)$, $\overline{N}(r, f) = N_{1)}(r, f) + \overline{N}_{(2}(r, f)$.

Finally we denote by $Z_{[2]}(r, f)$ the counting function of the zeroes of $f$ each counted multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

And here we denote by $Z_0(r, f')$ the counting function of the zeroes of $f'$ that are not zeroes of $f(f-1)$.

We will now prove the following Lemma 11 in a similar way as in [13], with however some special properties due to $p$-adic analytic functions:

**Lemma 11.** *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) be such that $f(0) \neq 0, \infty$, and share the value 1 C.M. If $\Psi_{f,g}$ is not identically zero, then,*

$$\max\bigl(T(r, f), T(r, g)\bigr) \leqslant N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) - 3\log r.$$

**Proof.** Since $f$ and $g$ share 1 C.M., each simple zero of $f-1$ is a simple zero of $g-1$ and is a zero of $\Psi_{f,g}$. Consequently, we have

$$\overline{Z}_{(2}(r, f-1) = \overline{Z}_{(2}(r, g-1) \tag{1}$$

and

$$Z_{1)}(r, f-1) = Z_{1)}(r, g-1) \leqslant Z(r, \Psi_{f,g}). \tag{2}$$

Now, by Lemma 4 we have $Z(r, \Psi_{f,g}) \leqslant N(r, \Psi_{f,g}) - \log r$. Hence by (2) we obtain

$$Z_{1)}(r, f-1) = Z_{1)}(r, g-1) \leqslant N(r, \Psi_{f,g}) - \log r. \tag{3}$$

On the other hand, all poles of $\Psi_{f,g}$ are simple and only occur at zeroes of $f'$ and $g'$ and at multiple poles of $f$ and $g$. Consequently, we have

$$N(r, \Psi_{f,g}) \leqslant \overline{Z}_0\bigl(r, f'\bigr) + \overline{Z}_0\bigl(r, g'\bigr) + \overline{Z}_{(2}(r, f) + \overline{Z}_{(2}(r, g). \tag{4}$$

By (3) and (4), we have

$$Z_{1)}(r, f-1) \leqslant \overline{Z}_0\bigl(r, f'\bigr) + \overline{Z}_0\bigl(r, g'\bigr) + \overline{Z}_{(2}(r, f) + \overline{Z}_{(2}(r, g) - \log r,$$

therefore

$$\begin{aligned} Z_{1)}(r, f-1) \leqslant{}& \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{Z}_0\bigl(r, f'\bigr) + \overline{Z}_0\bigl(r, g'\bigr) \\ &+ \overline{Z}_{(2}(r, f) + \overline{Z}_{(2}(r, g) - \log r. \end{aligned} \tag{5}$$

Recall that by Lemma 2 we have

$$Z\bigl(r, g'\bigr) \leqslant Z(r, g) + \overline{N}(r, g) - \log r \tag{6}$$

and from the definition of $Z_0(r, g')$ we have

$$\overline{Z}_0\bigl(r, g'\bigr) + \overline{Z}_{(2}(r, g-1) + Z_{(2}(r, g) - \overline{Z}_{(2}(r, g) \leqslant Z\bigl(r, g'\bigr),$$

consequently, we obtain

$$\overline{Z}_0\big(r, g'\big) + \overline{Z}_{(2}(r, g-1) \leqslant Z\big(r, g'\big) + \overline{Z}_{(2}(r, g) - Z_{(2}(r, g). \tag{7}$$

But by (6) and (7) we have

$$\overline{Z}_0\big(r, g'\big) + \overline{Z}_{(2}(r, g-1) \leqslant Z(r, g) + \overline{Z}_{(2}(r, g) - Z_{(2}(r, g) + \overline{N}(r, g) - \log r$$

and $Z(r, g) + \overline{Z}_{(2}(r, g) - Z_{(2}(r, g)$ is just $\overline{Z}(r, g)$. Consequently, by the last inequality, we have

$$\overline{Z}_0\big(r, g'\big) + \overline{Z}_{(2}(r, g-1) \leqslant \overline{N}(r, g) + \overline{Z}(r, g) - \log r. \tag{8}$$

Now, Theorem N lets us write

$$T(r, f) \leqslant \overline{N}(r, f) + \overline{Z}(r, f) + \overline{Z}(r, f-1) - Z_0\big(r, f'\big) - \log r. \tag{9}$$

By (2) we notice that $\overline{Z}(r, f-1) = Z_{1)}(r, f-1) + \overline{Z}_{(2}(r, f-1) = Z_{1)}(r, f-1) + \overline{Z}_{(2}(r, g-1)$. So, by (9) we have

$$T(r, f) \leqslant \overline{N}(r, f) + \overline{Z}(r, f) + Z_{1)}(r, f-1) + \overline{Z}_{(2}(r, g-1) - Z_0\big(r, f'\big) - \log r$$

and hence, by (8), we derive

$$\begin{aligned} T(r, f) \leqslant{} & \overline{N}(r, f) + \overline{Z}(r, f) + Z_{1)}(r, f-1) + \overline{N}(r, g) + \overline{Z}(r, g) - \overline{Z}_0\big(r, g'\big) \\ & - Z_0\big(r, f'\big) - 2\log r. \end{aligned}$$

And now, by (5) we obtain

$$\begin{aligned} T(r, f) \leqslant{} & \overline{N}(r, f) + \overline{Z}(r, f) + \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{Z}_{(2}(r, f) + \overline{Z}_{(2}(r, g) \\ & + \overline{Z}_0\big(r, g'\big) + \overline{Z}_0\big(r, f'\big) + \overline{N}(r, g) + \overline{Z}(r, g) - \overline{Z}_0\big(r, g'\big) - Z_0\big(r, f'\big) - 3\log r. \end{aligned}$$

But now, we notice that $\overline{Z}(r, f) + \overline{Z}_{(2}(r, f) = Z_{[2]}(r, f)$, $\overline{Z}(r, g) + \overline{Z}_{(2}(r, g) = Z_{[2]}(r, g)$, $\overline{N}(r, f) + \overline{N}_{(2}(r, f) = N_{[2]}(r, f)$, $\overline{N}(r, g) + \overline{N}_{(2}(r, g) = N_{[2]}(r, g)$. We then obtain the conclusion of Lemma 11. □

## 4. Proof of theorems

The polynomial $P$ is considered in Theorems 1, 2, 3, 4, 5, 6, 7 and we can assume $a_1 = 0$. In Theorems 8 and 9, we call $P$ the polynomial such that $P'(x) = x^n(x-a)^k$ and $P(0) = 0$. Set $F = \frac{f'P'(f)}{\alpha}$ and $G = \frac{g'P'(g)}{\alpha}$. Clearly $F$ and $G$ share the value 1 C.M.

Since $f$, $g$ are transcendental (resp. unbounded), we notice that so are $F$ and $G$. Recall that

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}.$$

We will prove that under the hypotheses of each theorem, $\Psi_{F,G}$ is identically zero.

Set $\widehat{F} = P(f)$, $\widehat{G} = P(g)$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in K[x]$ of degree $k$. Now, with help of Lemma 2, we can check that we have

$$T(r, \widehat{F}) - Z(r, \widehat{F}) \leqslant T\big(r, \widehat{F}'\big) - Z\big(r, \widehat{F}'\big) + O(1).$$

Consequently, since $(\widehat{F})' = \alpha F$, we have

$$T(r, \widehat{F}) \leqslant T(r, F) + Z(r, \widehat{F}) - Z(r, F) + T(r, \alpha) + O(1), \tag{1}$$

hence, by (1), we obtain

$$\begin{aligned} T(r, \widehat{F}) \leqslant\ & T(r, F) + (n+1)Z(r, f) + Z\big(r, Q(f)\big) - nZ(r, f) \\ & - \sum_{i=2}^{l} k_i Z(r, f - a_i) - Z\big(r, f'\big) + T(r, \alpha) + O(1), \end{aligned}$$

i.e.

$$\begin{aligned} T(r, \widehat{F}) \leqslant\ & T(r, F) + Z(r, f) + Z\big(r, Q(f)\big) \\ & - \sum_{i=2}^{l} k_i Z(r, f - a_i) - Z\big(r, f'\big) + T(r, \alpha) + O(1) \end{aligned} \tag{2}$$

and similarly,

$$\begin{aligned} T(r, \widehat{G}) \leqslant\ & T(r, G) + Z(r, g) + Z\big(r, Q(g)\big) \\ & - \sum_{i=2}^{l} k_i Z(r, g - a_i) - Z\big(r, g'\big) + T(r, \alpha) + O(1). \end{aligned} \tag{3}$$

Now, it follows from the definition of $F$ and $G$ that

$$\begin{aligned} Z_{[2]}(r, F) + N_{[2]}(r, F) \leqslant\ & 2Z(r, f) + 2\sum_{i=2}^{l} Z(r, f - a_i) + Z\big(r, f'\big) \\ & + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \end{aligned} \tag{4}$$

and similarly

$$\begin{aligned} Z_{[2]}(r, G) + N_{[2]}(r, G) \leqslant\ & 2Z(r, g) + 2\sum_{i=2}^{l} Z(r, g - a_i) + Z\big(r, g'\big) \\ & + 2\overline{N}(r, g) + T(r, \alpha) + O(1). \end{aligned} \tag{5}$$

And particularly, if $k_i = 1\ \forall i \in \{2, \dots, l\}$, then

$$\begin{aligned} Z_{[2]}(r, F) + N_{[2]}(r, F) \leqslant\ & 2Z(r, f) + \sum_{i=2}^{l} Z(r, f - a_i) + Z\big(r, f'\big) \\ & + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \end{aligned} \tag{6}$$

and similarly

$$\begin{aligned} Z_{[2]}(r, G) + N_{[2]}(r, G) \leqslant\ & 2Z(r, g) + \sum_{i=2}^{l} Z(r, g - a_i) + Z\big(r, g'\big) \\ & + 2\overline{N}(r, g) + T(r, \alpha) + O(1). \end{aligned} \tag{7}$$

Suppose now that $\Psi_{F,G}$ is not identically zero. Now, by Lemma 11, we have

$$T(r, F) \leqslant Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) - 3\log r$$

hence by (2), we obtain

$$T(r,\widehat{F}) \leqslant Z_{[2]}(r,F) + N_{[2]}(r,F) + Z_{[2]}(r,G) + N_{[2]}(r,G) + Z(r,f) + Z\big(r,Q(f)\big)$$
$$- \sum_{i=2}^{l} k_i Z(r,f-a_i) - Z\big(r,f'\big) + T(r,\alpha) - 3\log r + O(1)$$

and hence by (4) and (5):

$$\begin{aligned} T(r,\widehat{F}) \leqslant{}& 2Z(r,f) + 2\sum_{i=2}^{l} Z(r,f-a_i) + Z\big(r,f'\big) + 2\overline{N}(r,f) + 2Z(r,g) \\ &+ 2\sum_{i=2}^{l} Z(r,g-a_i) + Z\big(r,g'\big) + 2\overline{N}(r,g) + Z(r,f) + Z\big(r,Q(f)\big) \\ &- \sum_{i=2}^{l} k_i Z(r,f-a_i) - Z\big(r,f'\big) + 2T(r,\alpha) - 3\log r + O(1) \end{aligned} \tag{8}$$

and similarly,

$$\begin{aligned} T(r,\widehat{G}) \leqslant{}& 2Z(r,g) + 2\sum_{i=2}^{l} Z(r,g-a_i) + Z\big(r,g'\big) + 2\overline{N}(r,g) + 2Z(r,f) \\ &+ 2\sum_{i=2}^{l} Z(r,f-a_i) + Z\big(r,f'\big) + 2\overline{N}(r,f) + Z(r,g) + Z\big(r,Q(g)\big) \\ &- \sum_{i=2}^{l} k_i Z(r,g-a_i) - Z\big(r,g'\big) + 2T(r,\alpha) - 3\log r + O(1). \end{aligned} \tag{9}$$

Consequently,

$$\begin{aligned} T(r,\widehat{F}) + T(r,\widehat{G}) \leqslant{}& 5\big(Z(r,f) + Z(r,g)\big) + \sum_{i=2}^{l} (4-k_i)\big(Z(r,f-a_i) + Z(r,g-a_i)\big) \\ &+ \big(Z\big(r,f'\big) + Z\big(r,g'\big)\big) + 4\big(\overline{N}(r,f) + \overline{N}(r,g)\big) \\ &+ \big(Z\big(r,Q(f)\big) + Z\big(r,Q(g)\big)\big) + 4T(r,\alpha) - 6\log r + O(1). \end{aligned} \tag{10}$$

Moreover, if $k_i = 1$, $\forall i \in \{2,\dots,l\}$, then by (6) and (7) we have

$$\begin{aligned} T(r,\widehat{F}) \leqslant{}& 2Z(r,f) + \sum_{i=2}^{l} Z(r,f-a_i) + Z\big(r,f'\big) + 2\overline{N}(r,f) + 2Z(r,g) \\ &+ \sum_{i=2}^{l} Z(r,g-a_i) + Z\big(r,g'\big) + 2\overline{N}(r,g) + Z(r,f) + Z\big(r,Q(f)\big) \\ &- \sum_{i=2}^{l} Z(r,f-a_i) - Z\big(r,f'\big) + 2T(r,\alpha) - 3\log r + O(1) \end{aligned}$$

and similarly,

$$T(r,\widehat{G}) \leqslant 2Z(r,g) + \sum_{i=2}^{l} Z(r,g-a_i) + Z(r,g') + 2\overline{N}(r,g) + 2Z(r,f)$$
$$+ \sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f') + 2\overline{N}(r,f) + Z(r,g) + Z(r,Q(g))$$
$$- \sum_{i=2}^{l} Z(r,g-a_i) - Z(r,g') + 2T(r,\alpha) - 3\log r + O(1).$$

Consequently,

$$\begin{aligned}
&T(r,\widehat{F}) + T(r,\widehat{G})\\
&\quad\leqslant 5\big(Z(r,f)+Z(r,g)\big) + \sum_{i=2}^{l}\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\\
&\qquad + Z\big(r,Q(f)\big) + Z\big(r,Q(g)\big) + \big(Z(r,f') + Z(r,g')\big) + 4\big(\overline{N}(r,f)+\overline{N}(r,g)\big)\\
&\qquad + 4T(r,\alpha) - 6\log r + O(1).
\end{aligned} \tag{11}$$

Now, let us go back to the general case. By Lemma 2, we can write $Z(r,f') + Z(r,g') \leqslant Z(r,f-a_2) + Z(r,g-a_2) + \overline{N}(r,f) + \overline{N}(r,g) - 2\log r$. Hence, in general, by (10) we obtain

$$\begin{aligned}
&T(r,\widehat{F}) + T(r,\widehat{G})\\
&\quad\leqslant 5\big(Z(r,f)+Z(r,g)\big) + \sum_{i=3}^{l}(4-k_i)\big(\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\big)\\
&\qquad + (5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big) + 5\big(\overline{N}(r,f)+\overline{N}(r,g)\big)\\
&\qquad + \big(Z\big(r,Q(f)\big) + Z\big(r,Q(g)\big)\big) + 4T(r,\alpha) - 8\log r + O(1)
\end{aligned}$$

and hence, since $T(r,Q(f)) = kT(r,f) + O(1)$ and $T(r,Q(g)) = kT(r,g) + O(1)$,

$$\begin{aligned}
T(r,\widehat{F}) + T(r,\widehat{G}) &\leqslant 5\big(T(r,f)+T(r,g)\big) + \sum_{i=3}^{l}(4-k_i)\big(\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\big)\\
&\quad + (5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big) + 5\big(\overline{N}(r,f)+\overline{N}(r,g)\big)\\
&\quad + k\big(T(r,f)+T(r,g)\big) + 4T(r,\alpha) - 8\log r + O(1).
\end{aligned} \tag{12}$$

And if $k_i = 1$, $\forall i \in \{2,\dots,l\}$, by (11) and Lemma 2 we have

$$\begin{aligned}
T(r,\widehat{F}) + T(r,\widehat{G}) &\leqslant 5\big(Z(r,f)+Z(r,g)\big)\\
&\quad + \sum_{i=2}^{l}\big(Z(r,f-a_i)+Z(r,g-a_i)\big) + (l-1)\big(T(r,f)+T(r,g)\big)\\
&\quad + \big(Z(r,f-a_2)+Z(r,g-a_2)\big) + 5\big(\overline{N}(r,f)+\overline{N}(r,g)\big)\\
&\quad + 4T(r,\alpha) - 8\log r + O(1)
\end{aligned}$$

hence

$$\begin{aligned}T(r,\widehat{F})+T(r,\widehat{G}) \leqslant{}& 5\big(T(r,f)+T(r,g)\big)\\ &+\sum_{i=2}^{l}\big(T(r,f-a_i)+T(r,g-a_i)\big)+(l-1)\big(T(r,f)+T(r,g)\big)\\ &+\big(T(r,f-a_2)+T(r,g-a_2)\big)\\ &+5\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+4T(r,\alpha)-8\log r+O(1)\end{aligned}$$

and hence

$$T(r,\widehat{F})+T(r,\widehat{G}) \leqslant (9+2l)\big(T(r,f)+T(r,g)\big)+4T(r,\alpha)-8\log r+O(1). \tag{13}$$

Now, let us go back to the general case. Since $\widehat{F}$ is a polynomial in $f$ of degree $n+k+1$, we have $T(r,\widehat{F})=(n+k+1)T(r,f)+O(1)$ and similarly, $T(r,\widehat{G})=(n+k+1)T(r,g)+O(1)$, hence by (12) we can derive

$$\begin{aligned}&(n+k+1)\big(T(r,f)+T(r,g)\big)\\ &\quad\leqslant 5\big(T(r,f)+T(r,g)\big)+(5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)\\ &\qquad+\sum_{i=3}^{l}(4-k_i)\big(\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\big)\\ &\qquad+5\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+k\big(T(r,f)+T(r,g)\big)+4T(r,\alpha)-8\log r+O(1).\end{aligned} \tag{14}$$

Hence

$$\begin{aligned}&(n+k+1)\big(T(r,f)+T(r,g)\big)\\ &\quad\leqslant (10+k)\big(T(r,f)+T(r,g)\big)+\sum_{i=3}^{l}(4-k_i)\big(\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\big)\\ &\qquad+(5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)+4T(r,\alpha)-8\log r+O(1)\end{aligned}$$

and hence

$$\begin{aligned}&n\big(T(r,f)+T(r,g)\big)\\ &\quad\leqslant 9\big(T(r,f)+T(r,g)\big)+\sum_{i=3}^{l}(4-k_i)\big(\big(Z(r,f-a_i)+Z(r,g-a_i)\big)\big)\\ &\qquad+(5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)+4T(r,\alpha)-8\log r+O(1).\end{aligned} \tag{15}$$

Then at least, for each $i=3,\dots,l$ we have

$$(4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big) \leqslant \max(0,4-k_i)\big(T(r,f)+T(r,g)\big)+O(1)$$

and

$$(5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big) \leqslant \max(0,5-k_2)\big(T(r,f)+T(r,g)\big)+O(1).$$

Consequently, by (15) we have

$$\begin{aligned}n\big(T(r,f)+T(r,g)\big) \leqslant{}& 9\big(T(r,f)+T(r,g)\big)+\sum_{i=3}^{l}\max(0,4-k_i)\big(T(r,f)+T(r,g)\big)\\ &+\max(0,5-k_2)\big(T(r,f)+T(r,g)\big)+O(1)\end{aligned}$$

and hence,

$$n \leqslant 9 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2). \tag{16}$$

Moreover, if $f$, $g$ belong to $\mathcal{M}(\mathbb{K})$ and $\alpha$ is a constant or a Moebius function, then $T(r,\alpha) \leqslant \log r + O(1)$ and hence by (15 )we have

$$n \leqslant 8 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2). \tag{17}$$

Now, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, by (13) we have $n + k + 1 \leqslant 9 + 2l$, hence

$$n \leqslant 9 + l. \tag{18}$$

Particularly, if $f, g \in \mathcal{M}(\mathbb{K})$ and if $\alpha$ is a constant or a Moebius function, then

$$n \leqslant 8 + l. \tag{19}$$

Consequently, in the general case, if

$$n \geqslant 10 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$$

we have $\Psi_{F,G} = 0$ which concerns Theorems 1 and 4.

Now, if $f$, $g$ belong to $\mathcal{M}(\mathbb{K})$ and $\alpha$ is a constant or a Moebius function and if

$$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$$

we have $\Psi_{F,G} = 0$ again, which concerns Theorems 2 and 3.

Further, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, when $n \geqslant l + 10$ we have $\Psi_{F,G} = 0$ which concerns Theorems 5, 6, 10 and 9 when $\alpha$ is an ordinary small function.

In the same context, if $f$, $g$ belong to $\mathcal{M}(\mathbb{K})$, and if $\alpha$ is a constant or a Moebius function, then $\Psi_{F,G} = 0$ as soon as $n \geqslant l + 9$ Theorems 7, 8 and 9 when $\alpha$ is a constant or a Moebius function.

Thus, henceforth, we can assume that $\Psi_{F,G} = 0$ in each hypothesis of all theorems.

Note that $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = (\frac{F'}{(F-1)^2})(\frac{(G-1)^2}{G'})$. Since $\phi' = 0$, there exist $A, B \in \mathbb{K}$ such that

$$\frac{1}{G-1} = \frac{A}{F-1} + B \tag{20}$$

and $A \neq 0$.

Now, we will consider the following two cases: $B = 0$ and $B \neq 0$.

**Case 1.** $B = 0$.

Suppose $A \neq 1$. Then, by (20), we have $F = \frac{1}{A}G + (1 - \frac{1}{A})$. Applying Theorem N to $F$, we obtain

$$\begin{aligned} T(r,F) &\leqslant \overline{Z}(r,F) + \overline{Z}\left(r, F - \left(1 - \frac{1}{A}\right)\right) + \overline{N}(r,F) - \log r + O(1) \\ &\leqslant \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{Z}\big(r, f'\big) + \overline{Z}(r,g) \\ &\quad + \sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}\big(r, g'\big) + \overline{N}(r,f) + 3T(r,\alpha) - \log r + O(1). \end{aligned} \tag{21}$$

But $\overline{Z}(r,f) \leqslant T(r,f)$, $\overline{N}(r,f) \leqslant T(r,f)$, $\overline{Z}(r, f-1) \leqslant T(r, f-1) \leqslant T(r,f) + O(1)$ and $\overline{Z}(r,f') \leqslant T(r,f') \leqslant 2T(r,f) + O(1)$. Moreover, by Lemma 1, $T(r,F) \geqslant (n+k)T(r,f) - T(r,\alpha)$. Then, considering all the previous inequalities in (12), we deduce that

$$(n+k)T(r,f) \leqslant (l+3)T(r,f) + (l+2)T(r,g) + 4T(r,\alpha) - \log r + O(1). \tag{22}$$

Since $f$ and $g$ satisfy the same hypothesis, we also have

$$(n+k)T(r,g) \leqslant (l+3)T(r,g) + (l+2)T(r,f) + 4T(r,\alpha) - \log r + O(1). \tag{23}$$

Hence, adding (22) and (23), we have

$$(n+k)\big[T(r,f) + T(r,g)\big] \leqslant (2l+5)\big[T(r,f) + T(r,g)\big] + 4T(r,\alpha) + O(1),$$

which leads to a contradiction whenever $n + k \geqslant (2l + 6)$.

In the hypotheses of all theorems we have

$$n \geqslant 9 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2).$$

That implies

$$\begin{aligned} n + k &\geqslant 9 + k + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2) \\ &= 9 + \sum_{i=2}^{l} k_i + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2) \\ &= 9 + \sum_{i=3}^{l} \max(4, k_i) + \max(5, k_2) \geqslant 9 + 4(l-2) + 5 \geqslant 4l + 6. \end{aligned}$$

Consequently, the inequality $n + k \geqslant (2l + 6)$ is satisfied in Theorems 1, 2, 3, 4.

Next, if all $k_i$ are equal to 1, we assume that $n \geqslant l + 9$, hence $n + k \geqslant l + k + 9 = 2l + 8$ (because $l = k + 1$) and hence the inequality $n + k \geqslant (2l + 6)$ is satisfied again in Theorems 5, 6, 7, 8, 9, 10.

Hence, we have $A = 1$ and this implies that $F = G$. Now, $\alpha F = \alpha G$, i.e. $(\widehat{F})' = (\widehat{G})'$. We assume $n \geqslant k + 2$ in Theorems 1, 2, 3 and this is automatically satisfied in Theorems 5, 7, 8, 9. And we assume $n \geqslant k + 3$ in Theorem 4 and this is automatically satisfied in Theorems 6 and 10. Consequently, by Lemma 7, we have $\widehat{F} = \widehat{G}$, i.e. $P(f) = P(g)$. But in Theorems 1, 2, 3, 4, 5, 6, 7, 8 $P$ is a polynomial of uniqueness for the family of meromorphic functions we consider, hence we have $f = g$. And in Theorems 9 and 10, the conclusion comes from Lemma 10.

**Case 2.** $B \neq 0$.

We have $\overline{Z}(r, F) \leqslant \overline{Z}(r, f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + T(r, \alpha)$ and $\overline{N}(r, F) \leqslant \overline{N}(r, f) + T(r, \alpha) + O(1)$ and similarly for $G$, so we can derive

$$\begin{aligned}
&\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \\
&\quad \leqslant \overline{Z}(r, f) + \sum_{i=2}^{l} \overline{Z}(r, f - a_i) + \overline{Z}\big(r, f'\big) + \overline{Z}(r, g) \\
&\qquad + \sum_{i=2}^{l} \overline{Z}(r, g - a_i) + \overline{Z}\big(r, g'\big) + \overline{N}(r, f) + \overline{N}(r, g) + 4T(r, \alpha) + O(1) \\
&\quad \leqslant 5\big[T(r, f) + T(r, g)\big] + 4T(r, \alpha) + O(1).
\end{aligned} \tag{24}$$

Moreover, by (20), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 1, we have $T(r, f) \leqslant \frac{1}{n+k}(T(r, F) + T(r, \alpha)) + O(1)$ and $T(r, g) \leqslant \frac{1}{n+k}(T(r, G) + T(r, \alpha)) + O(1)$. Consequently,

$$T(r, f) + T(r, g) \leqslant 2\left[\frac{1}{n+k}\big(T(r, F) + T(r, \alpha)\big)\right] + O(1).$$

Thus, (24) is equivalent to

$$\begin{aligned}
&\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \\
&\quad \leqslant \frac{10}{n+k} T(r, F) + \left(\frac{10}{n+k} + 4\right) T(r, \alpha) + O(1).
\end{aligned}$$

Now, we can check that $n + k \geqslant 12$ in all theorems. Consequently, the previous inequality implies

$$\begin{aligned}
&\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \\
&\quad \leqslant \frac{10}{12} T(r, F) + \left(\frac{10}{12} + 4\right) T(r, \alpha) + O(1).
\end{aligned} \tag{25}$$

Consequently, by (25) we can see that the hypotheses of Lemma 8 are satisfied and hence, either $F = G$, or $FG = 1$.

If $FG = 1$, then $f'P'(f)g'P'(g) = \alpha^2$. In Theorems 1, 2, 4 we have assumed that $n \geqslant k + 2$ and if $l = 2$, then $n \neq 2k, 2k + 1, 3k + 1$ and if $l = 3$ then $n \neq 3k_2 - k, 3k_3 - k$. Moreover, these conditions are automatically satisfied in Theorems 5, 6, 7, 8, 9, 10, so we have a contradiction to Lemma 9. In Theorem 3, we have assumed that $n \geqslant k + 2$ hence by Lemma 9, we have a contradiction again. Consequently, $F = G$, hence $(\widehat{F})' = (\widehat{G})'$ and therefore we can conclude as in the case $B = 0$.

## References


[1] T.T.H. An, J.T.Y. Wang, P.M. Wong, Unique range sets and uniqueness polynomials in positive characteristic II, Acta Arith. 116 (2005) 115–143.

[2] A. Boutabaa, Théorie de Nevanlinna $p$-adique, Manuscripta Math. 67 (1990) 251–269.

[3] A. Boutabaa, A. Escassut, URS and URSIMS for $p$-adic meromorphic functions inside a disc, Proc. Edinb. Math. Soc. 44 (2001) 485–504.

[4] A. Escassut, L. Haddad, R. Vidal, Urs, Ursim and nonurs, J. Number Theory 75 (1999) 133–144.

[5] A. Escassut, Meromorphic functions of uniqueness, Bull. Sci. Math. 131 (3) (2007) 219–241.

[6] A. Escassut, J. Ojeda, C.C. Yang, Functional equations in a $p$-adic context, J. Math. Anal. Appl. 351 (1) (2009) 350–359.

[7] M. Fang, X.H. Hua, Entire functions that share one value, J. Nanjing Univ. Math. Biq. 13 (1) (1996) 44–48.
[8] M. Fang, W. Hong, A unicity theorem for entire functions concerning differential polynomials, Indian J. Pure Appl. Math. 32 (9) (2001) 1343–1348.
[9] G. Frank, M. Reinders, A unique range set for meromorphic functions with 11 elements, Complex Variable Theory Appl. 37 (1998) 185–193.
[10] H. Fujimoto, On uniqueness of meromorphic functions sharing finite sets, Amer. J. Math. 122 (6) (2000) 1175–1203.
[11] N.T. Hoa, On the functional equation $P(f) = Q(g)$ in non-archimedean field, Acta Math. Vietnam. 31 (2) (2006) 167–180.
[12] P.C. Hu, C.C. Yang, Meromorphic Functions over Non-Archimedean Fields, Kluwer Academy Publishers, 2000.
[13] X. Hua, C.C. Yang, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math. 22 (1997) 395–406.
[14] H.H. Khoai, T.T.H. An, On uniqueness polynomials and bi-URs for $p$-adic meromorphic functions, J. Number Theory 87 (2) (2001) 211–221.
[15] I. Lahiri, N. Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, Int. J. Math. Math. Sci. 12 (2005) 1933–1942.
[16] P. Li, C.C. Yang, Some further results on the unique range sets of meromorphic functions, Kodai Math. J. 18 (1995) 437–450.
[17] W. Lin, H. Yi, Uniqueness theorems for meromorphic functions concerning fixed-points, Complex Var. Theory Appl. 49 (11) (2004) 793–806.
[18] J. Ojeda, Applications of the $p$-adic Nevanlinna theory to problems of uniqueness, advances in $p$-adic and non-Archimedean analysis, Contemporary Mathematics 508 (2010) 161–179.
[19] J. Ojeda, Zeros of ultrametric meromorphic functions $f'f^n(f-a)^k-\alpha$, Asian-Eur. J. Math. 1 (3) (2008) 415–429.
[20] J. Ojeda, Uniqueness for ultrametric analytic functions, Bull. Math. Sci. Math. Roumanie, submitted for publication.
[21] J.T.Y. Wang, Uniqueness polynomials and bi-unique range sets, Acta Arith. 104 (2002) 183–200.
[22] Y. Xu, H. Qu, Entire functions sharing one value I.M., Indian J. Pure Appl. Math. 31 (7) (2000) 849–855.
[23] C.C. Yang, X. Hua, Unique polynomials of entire and meromorphic functions, Mat. Fiz. Anal. Geom. 4 (3) (1997) 391–398.
[24] H.X. Yi, C.C. Yang, Uniqueness Theorems of Meromorphic Functions, Science Press, China, 1995.